Exercise (1)

Use power series to solve the equation

(1) $y'' + y = 0$ (2) $y'' - 2xy' + y = 0$

(3) $y' - y = 0$ (4) $y' - xy = 0$

(5) $y' - x^2 y = 0$ (6) $(x-3)y' + 2y = 0$

(7) $y'' + xy' + y = 0$ (8) $y'' = y$

(9) $y'' - xy = 0$ (10) $y'' - xy' - y = 0, \quad y(0) = 1, y'(0) = 0$

(11) $y'' + x^2 y = 0$ (12) $y'' + x^2 y' + xy = 0, y(0) = 0, y'(0) = 1$

Answers

(1) $y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(2) $y = a_0 \left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3.7.11...(4n-5)}{(2n)!} x^{2n} \right) + a_1 \left(x + \sum_{n=1}^{\infty} \frac{1.5.9.13...(4n-3)}{(2n+1)!} x^{2n+1} \right).$

(3) $y = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x$ (4) $y = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n.(n)!}$

(5) $y = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!}$ (6) $y == c_0 \left(1 + \frac{2}{3}x + \frac{1}{3}x^2 + \frac{4}{27}x^3 + \frac{5}{81}x^4 + ... \right)$

(7) $y = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n (n)!} x^{2n} \right) + a_1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n 2^n \text{ n!}}{(2n+1)!} x^{2n+1} \right)$

(8) $y = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$

(9) $y = c_0 \left[1 + \frac{x^3}{3!} + \frac{4x^6}{6!} + ... \right] + c_1 \left[x + \frac{2x^4}{4!} + \frac{10x^7}{8!} + ... \right]$

(10) $y = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

(11) $y = a_0 \left(1 - \frac{1}{12}x^4 + \frac{1}{672} x^8 - ... \right) + a_1 \left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 + ... \right)$

(12) $y = c_1 \left(x - \frac{1}{6}x^4 + \frac{5}{252}x^7 + ... \right)$

Answers of Text Book Exercise
Page 49

(1) $4x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-y=0$

Answer $y_1(x)=1+\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!1.5.9...(4n-3)}x^n$ *and* $y_2(x)=x^{3/4}\left\{1+\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!7.11.15...(4n+3)}x^n\right\}$

(2) $x\dfrac{d^2y}{dx^2}+(x+1)\dfrac{dy}{dx}+y=0$

Answer $y_1(x)=e^{-x}$ *and* $y_2(x)=\ln x.e^{-x}-\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)x^n$

(3) $x\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=0$

Answer $y_1(x)=\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{1}{2n!(n-3)!}x^n$

$$y_2(x)=y_1(x)\ln x+1+\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{36}x^3$$

$$+\sum_{n=3}^{\infty}(-1)^n\frac{1}{2n!(n-3)!}\left\{-2\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n-3}-\frac{1}{n-2}-\frac{1}{n-1}-\frac{1}{n}+\frac{3}{2}\right)\right\}x^n$$

(4) $9x(1-x)\dfrac{d^2y}{dx^2}-12\dfrac{dy}{dx}+4y=0$

Answer $y_1(x)=1+\displaystyle\sum_{n=1}^{\infty}\frac{1.4.7....(3n-2)}{3^n n!}x^n=(1-x)^{-1/2}$

$$y_2(x)=x^{7/3}\left\{1+\sum_{n=1}^{\infty}\frac{8.11.14...(3n+5)}{10.13.16...(3n+7)(4n+3)}x^n\right\},\quad |x|<1$$

(5) $(1-x^2)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+y=0$

Answer

$$y=a_0\left[1+\frac{-1}{(2)(1)}x^2+\frac{(-3)(-1)}{4.3.2.1}x^4+\frac{(-3)(-1).1}{6.5.4.3.2.1}x^6+...\right]$$

$$a_1\left[x+\frac{(-3)}{3.2.1}x^3+\frac{(-3)(-1)}{5.4.3.2.1}x^5+\frac{(-1)(-3)(9)}{7.6.5.4.3.2.1}x^7+...\right]$$

(6) $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$

Answer $y=a_1x+a_0\left[1+\frac{1}{2}x^2+\frac{1}{4!}x^4+\sum_{n=3}^{\infty}\frac{3^2.5^2.7^2...(2n-3)^2}{2n!}x^{2n}\right],\quad |x|<1$

(7) $\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}-y=0$

Answer

$$y=a_0\left[1+\tfrac{1}{2}x^2+\tfrac{1}{24}x^4+\tfrac{1}{20}x^5+\tfrac{1}{720}x^6+...\right]$$

$$+a_1\left[x+\tfrac{1}{6}x^3+\tfrac{1}{12}x^4+\tfrac{1}{120}x^5++\tfrac{7}{360}x^6+...\right]$$

valid for all x

(8) $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$

$$y=a_0+a_1x+\frac{a_0}{2}x^2+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-3)!}{2^{2n-2}n!(n-2)!}x^{2n}$$